



Chapter 2 : Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Lecture 7:

1. Set Operations , Set Identities, Computer Representation of Sets
2. Functions, one to one and onto functions

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2.2

Set Operations

DEFINITION 1

Let A and B be sets. The *union* of the sets A and B , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B , or in both.


An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B . This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

The Venn diagram shown in Figure 1 represents the union of two sets A and B . The area that represents $A \cup B$ is the shaded area within either the circle representing A or the circle representing B .

We will give some examples of the union of sets.

EXAMPLE 1

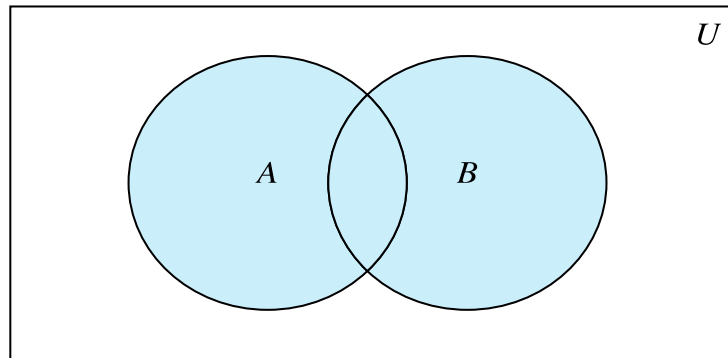
The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$. 

DEFINITION 2

Let A and B be sets. The *intersection* of the sets A and B , denoted by $A \cap B$, is the set containing those elements in both A and B .

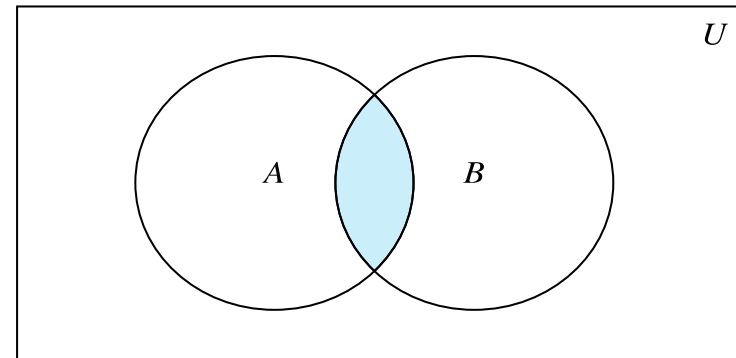
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B . This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram of the Union of A and B .



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram of the Intersection of A and B .

EXAMPLE 3 The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

DEFINITION 3 Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE 5 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint. ◀

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

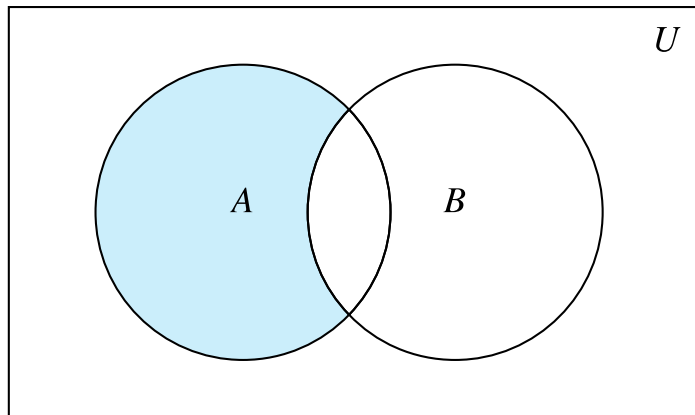
DEFINITION 4 Let A and B be sets. The *difference* of A and B , denoted by $A - B$, is the set containing those elements that are in A but not in B . The difference of A and B is also called the *complement of B with respect to A* .

Remark: The difference of sets A and B is sometimes denoted by $A \setminus B$.

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

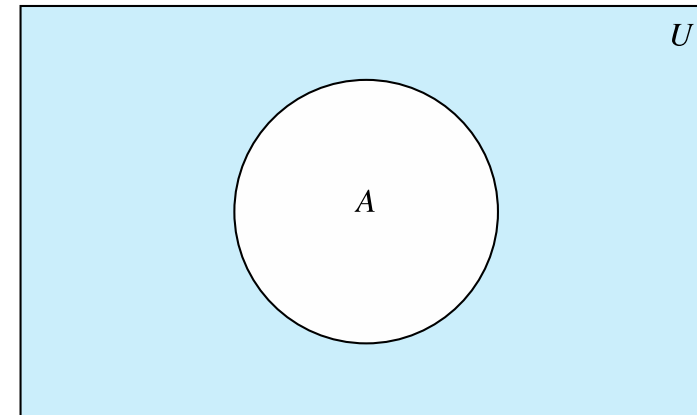
$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

EXAMPLE 6 The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ▶



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B .



\bar{A} is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A .

DEFINITION 5


Let U be the universal set. The *complement* of the set A , denoted by \bar{A} , is the complement of A with respect to U . Therefore, the complement of the set A is $U - A$.


An element belongs to \bar{A} if and only if $x \notin A$. This tells us that

$$\bar{A} = \{x \in U \mid x \notin A\}.$$

In Figure 4 the shaded area outside the circle representing A is the area representing \bar{A} .

We give some examples of the complement of a set.

EXAMPLE 8 Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. 

EXAMPLE 9 Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. 

It is left to the reader (Exercise 19) to show that we can express the difference of A and B as the intersection of A and the complement of B . That is,

$$A - B = A \cap \bar{B}.$$

Set Identities

TABLE 1 Set Identities.

<i>Identity</i>	<i>Name</i>
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities

EXAMPLE 10 Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.


$$\begin{aligned}\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\ &= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\ &= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\ &= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\ &= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\ &= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\ &= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\ &= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}\end{aligned}$$

EXAMPLE 12

Prove the second distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A , B , and C .

Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$. We conclude that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. 

EXAMPLE 13 Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

EXAMPLE 14 Let A , B , and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned} \overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{by the first De Morgan law} \\ &= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\ &= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\ &= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.} \end{aligned}$$

Generalized Unions and Intersections

DEFINITION 6

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^n A_i$$

to denote the union of the sets A_1, A_2, \dots, A_n .

DEFINITION 7

The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^n A_i$$

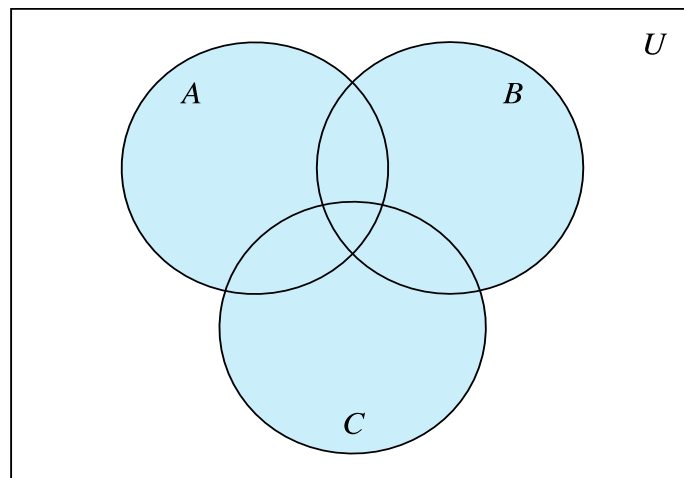
EXAMPLE 15 Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Solution: The set $A \cup B \cup C$ contains those elements in at least one of A , B , and C . Hence,

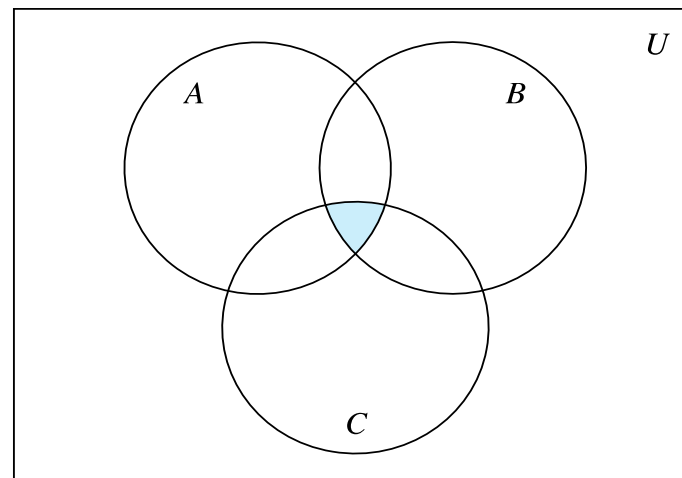
$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of A , B , and C . Thus,

$$A \cap B \cap C = \{0\}.$$



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

FIGURE 5 The Union and Intersection of A , B , and C .

Computer Representation of Sets

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U , for instance a_1, a_2, \dots, a_n . Represent a subset A of U with the bit string of length n , where the i th bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A . Example 18 illustrates this technique.

EXAMPLE 18 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U , the subset of all even integers in U , and the subset of integers not exceeding 5 in U ?

Solution: The bit string that represents the set of odd integers in U , namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading.) Similarly, we represent the subset of all even integers in U , namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000.




EXAMPLE 19 We have seen that the bit string for the set $\{1, 3, 5, 7, 9\}$ (with universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$) is

10 1010 1010.

What is the bit string for the complement of this set?

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set $\{2, 4, 6, 8, 10\}$. 


EXAMPLE 20 The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$. The bit string for the intersection of these sets is

$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$

which corresponds to the set $\{1, 3, 5\}$. 

2.3 Functions

DEFINITION 1

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A . We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A . If f is a function from A to B , we write $f : A \rightarrow B$.

Remark: Functions are sometimes also called **mappings** or **transformations**.

DEFINITION 2

If f is a function from A to B , we say that A is the *domain* of f and B is the *codomain* of f . If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b . The *range*, or *image*, of f is the set of all images of elements of A . Also, if f is a function from A to B , we say that f *maps* A to B .

Figure 2 represents a function f from A to B .

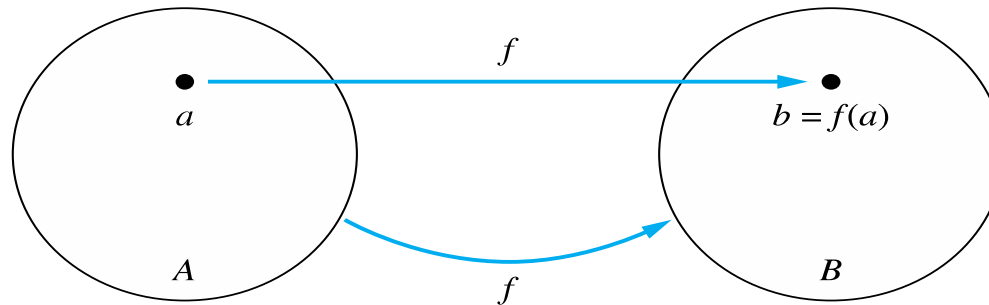



FIGURE 2 The Function f Maps A to B .

EXAMPLE 4 Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \dots\}$. 

DEFINITION 3

Let f_1 and f_2 be functions from A to \mathbf{R} . Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to \mathbf{R} defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

EXAMPLE 6 Let f_1 and f_2 be functions from \mathbf{R} to \mathbf{R} such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$



DEFINITION 4

Let f be a function from A to B and let S be a subset of A . The *image* of S under the function f is the subset of B that consists of the images of the elements of S . We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

EXAMPLE 7

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$. ◀

One-to-One and Onto Functions

DEFINITION 5

A function f is said to be *one-to-one*, or an *injection*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f . A function is said to be *injective* if it is one-to-one.

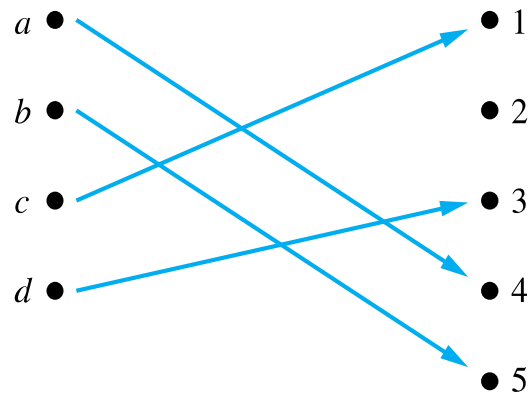


FIGURE 3 A One-to-One Function.

Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.

EXAMPLE 9 Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.